

Ricci flat metrics in various dimensions, depending from 2 light-cone parameters, and the Lagrangian for the 2 dimensional reduction of gravity

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Abstract

We consider d -dimensional Riemannian manifolds which admit $d - 2$ commuting space-like Killing vector fields, orthogonal to a surface, containing two one-parametric families of light-like curves. The condition of the Ricci tensor to be zero gives Ernst equations for the metric. We write explicitly a family of local solutions of this equations corresponding to arbitrary initial data on two characteristics in terms of a series. These metrics describe scattering of 2 gravitational waves, and thus we expect they are very interesting.

Ernst equations can be written as equations of motion for some 2D Lagrangian, which governs fluctuations of the metric, constant in the Killing directions. This Lagrangian looks essentially as a 2D chiral field model, and thus is possibly treatable in the quantum case by standart methods. It is conceivable that it may describe physics of some specially arranged scattering experiment, thus giving an insight for 4D gravity, not treatable by standart quantum field theory methods.

The renormalization flow for our Lagrangian is different from the flow for the unitary chiral field model, the difference is essentially due to the fact that here the field is taking values in a non-compact space of symmetric matrices. We investigate the model and derive the renormalized action in one loop.

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1 Introduction

Exact solutions of Einstein equations were of considerable interest for a long time and have important applications, say for cosmology; they also serve as backgrounds for quantum field theories, and they give an approximate solution for metrics in the target space for conformally invariant string theories.

There were many interesting approaches to get solutions of Einstein equations, most of them using symmetries, or Killing vectors. One of the most interesting discovery was the existence of an infinite-dimensional group of transformations of metrics, mapping a solution of Einstein equation to another solution (Geroch group).

If we have 2 commuting Killing vector fields in 4 dimensions, we can use them to reduce the problem to two dimensions. With some additional conditions on the Killing vectors satisfied, the free Einstein equations can be written in a nice form of Ernst equations, which admit a Lax pair representation as a compatibility condition of two auxiliary linear equations with a spectral parameter [7], [8], [9].

In the paper, we will write explicitly the local solutions of Ernst equation. We have effectively all such solutions, since we can satisfy arbitrary initial-boundary data. The solution is in term of a series. It is a very interesting problem to understand what are the properties of these solutions and what kind of singularities they develop.

If we use our Killing vectors to reduce the problem to two dimensions, we get an interesting Lagrangian in 2D, describing fluctuations of the metric, which are constant in the direction of our Killing vector fields. This Lagrangian is quite close to a principal chiral field model, but for fields with values in symmetric matrices. If we take this Lagrangian, but for fields with values in the $SU(n)$ group instead, (with $n = 2$ for reduction from 4 dimensions), we get just the principal chiral field, which admit an exact solution in the quantum case. We expect that this Lagrangian is treatable by standard methods, and that, unlike the unitary chiral field, it's not asymptotically free. We discuss the renormalization flow for this Lagrangian in one loop in the section 3.

2 Solutions of the Einstein Equation in Various Dimensions, Depending from 2 Light-Cone Parameters.

We consider Riemanian manifolds of dimension $d = 3, 4, \dots 6, \dots 10, 11, 12, \dots$ such that

1) they admit $(d - 2)$ space-like commuting Killing vectors fields, orthogonal to a (u, v) surface, which contains two one-parametric families of light-like curves. Thus it is possible to introduce coordinates $x = (u, v, x^2, x^3, \dots x^d)$, such that in that coordinates

$$ds^2 \equiv \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu = g_{ab}(u, v)dx^a dx^b - 2f(u, v)dudv, \quad (1)$$

$$a, b = 2, 3, \dots d.$$

2) The metric $\tilde{g}_{\mu\nu}(x)$ is Ricci flat,

$$R_{\mu\nu} = 0, \quad (2)$$

where $R_{\mu\nu}$ is the Ricci tensor.

Let us introduce a function $\varphi(u, v)$ as follows:

$$f(u, v) = G^{-\frac{1}{2}}(u, v) \exp(2\varphi(u, v)), \quad (3)$$

where

$$G(u, v) = \det(g_{ab}(u, v)).$$

The Ricci flatness condition yeilds the following equations:

$$R_{mn} = 0, \quad m, n = 2, 3 \dots d \Rightarrow \quad (4)$$

$$\left(G^{\frac{1}{2}}g_u g^{-1}\right)_v + \left(G^{\frac{1}{2}}g_v g^{-1}\right)_u = 0$$

Taking Trace in (4), we obtain

$$\left(G^{\frac{1}{2}}\right)_{uv} = 0 \quad (5)$$

The other nontrivial equations are

$$\begin{aligned}
R_{uu} = 0 \Rightarrow \\
\varphi_u = -\frac{1}{2} \frac{\left(G^{\frac{1}{2}}\right)_{uu}}{\left(G^{\frac{1}{2}}\right)_u} - \frac{1}{8} \frac{G^{\frac{1}{2}}}{\left(G^{\frac{1}{2}}\right)_u} \text{Tr} \left(g_u g^{-1} g_u g^{-1} \right) \\
R_{vv} = 0 \Rightarrow \\
\varphi_v = -\frac{1}{2} \frac{\left(G^{\frac{1}{2}}\right)_{vv}}{\left(G^{\frac{1}{2}}\right)_v} - \frac{1}{8} \frac{G^{\frac{1}{2}}}{\left(G^{\frac{1}{2}}\right)_v} \text{Tr} \left(g_v g^{-1} g_v g^{-1} \right),
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
R_{uv} = 0 \Rightarrow \\
\varphi_{uv} = \frac{1}{8} \text{Tr} \left(g_u g^{-1} g_v g^{-1} \right).
\end{aligned} \tag{7}$$

Using (4), (5), we can show that the equations (6) for φ are compatible,

$$\varphi_{uv} = \varphi_{vu} = \frac{1}{8} \text{Tr} \left(g_u g^{-1} g_v g^{-1} \right),$$

which coincides with (7). If we are given g , which is a solution of (4), then (6) is a compatible system of linear equations for φ , which can be easily integrated. Therefore, we need to solve only equations (4).

From (5) it follows that

$$a(u, v) \equiv G^{\frac{1}{2}}(u, v) = y(v) - x(u) \tag{8}$$

Let us choose the local coordinates x and y instead of u, v . In the new coordinates, the equations (4) become

$$\begin{aligned}
((y-x)g_x g^{-1})_y + ((y-x)g_y g^{-1})_x = 0, \quad -1 \leq x < y \leq 1 \\
g(-1, y) \text{ and } g(x, 1) \text{ are given.}
\end{aligned} \tag{9}$$

2.1 Linear Ernst Equation and Abel Transform.

Let $g(x, y)$ be a diagonal matrix,

$$g(x, y) = \begin{pmatrix} e^{\gamma^{(1)}(x, y)} & 0 \\ 0 & e^{\gamma^{(2)}(x, y)} \end{pmatrix}$$

In that case, the equation (9) reduces to a pair of linear equations of the form

$$2(y-x)\gamma_{xy} + \gamma_x - \gamma_y = 0, \quad i = 1, 2. \quad (10)$$

Proposition 1 *The solution of (10) is given by*

$$\begin{aligned} \gamma(x, y) &= \frac{1}{\pi i} \int_{-1}^x \frac{(\lambda - 1)^{\frac{1}{2}}}{(\lambda - x)^{\frac{1}{2}}(\lambda - y)^{\frac{1}{2}}} \hat{\gamma}_1(\lambda) d\lambda - \frac{1}{\pi i} \int_y^1 \frac{(\lambda + 1)^{\frac{1}{2}}}{(\lambda - x)^{\frac{1}{2}}(\lambda - y)^{\frac{1}{2}}} \hat{\gamma}_2(\lambda) d\lambda \\ \hat{\gamma}_1(\lambda) &= \int_{-1}^{\lambda} \frac{\gamma_x(x, 1)}{(\lambda - x)^{\frac{1}{2}}} dx, \quad \hat{\gamma}_2(\lambda) = \int_{\lambda}^1 \frac{\gamma_y(-1, y)}{(\lambda - y)^{\frac{1}{2}}} dy \end{aligned} \quad (11)$$

Remark 1 *Let us take $y = 1$. Then we will obtain a transformation*

$$\begin{aligned} \gamma(x) &= \frac{1}{\pi i} \int_{-1}^x \frac{1}{(\lambda - x)^{\frac{1}{2}}} \hat{\gamma}_1(\lambda) d\lambda \\ \hat{\gamma}_1(\lambda) &= \int_{-1}^{\lambda} \frac{\gamma_x(x, 1)}{(\lambda - x)^{\frac{1}{2}}} dx \end{aligned} \quad (12)$$

Such transformation is an Abel transform.

2.2 The Nonlinear Ernst Equation

We need $g(x, y)$ to be a real symmetric matrix, with the determinant equal to $(x - y)^2$. We first ignore these conditions and solve the equation for a general $n \times n$ complex matrix; then we will show how to satisfy the additional conditions.

Theorem. *Consider the Ernst equation*

$$((y - x)g_x g^{-1})_y + ((y - x)g_y g^{-1})_x = 0, \quad (13)$$

where $g(x, y)$ is a complex $n \times n$ matrix.

The following series,

$$\begin{aligned}
g(x, y) = & \left(I - 2 \int \frac{J(\lambda_1)}{p(\lambda_1)} d\lambda_1 + \right. \\
& + 2 \sum_{n=2}^{\infty} \int (0, \quad I) Z^{-}_{n,n-1} Z^{-}_{n-1,n-2} \dots Z^{-}_{2,1} \begin{pmatrix} -\frac{F(\lambda_1)}{p(\lambda_1)} \\ \frac{J(\lambda_1)}{p(\lambda_1)} \\ -\frac{J(\lambda_1)}{p(\lambda_1)} \end{pmatrix} \\
& \left. d\lambda_1 d\lambda_2 \dots d\lambda_n \right) g_0,
\end{aligned} \tag{14}$$

where

$$p(\lambda) := (\lambda - x)^{\frac{1}{2}} (\lambda - y)^{\frac{1}{2}}$$

$$Z^{\mp}_{nm} \equiv Z^{\mp}(\lambda_n, \lambda_m) = \begin{pmatrix} \frac{F(\lambda_n)}{\lambda_n - \lambda_m} \frac{p(\lambda_m)}{p(\lambda_n)}, & \frac{J(\lambda_n)}{\lambda_n - \lambda_m} \mp \frac{F(\lambda_n)}{p(\lambda_n)} \\ \frac{J(\lambda_n)}{\lambda_n - \lambda_m} \frac{p(\lambda_m)}{p(\lambda_n)}, & \frac{F(\lambda_n)}{\lambda_n - \lambda_m} \mp \frac{J(\lambda_n)}{p(\lambda_n)} \end{pmatrix}$$

$F(\lambda)$ is an arbitrary C^{∞} $n \times n$ matrix ,

$$J(\lambda) = \begin{cases} \text{an arbitrary } C^{\infty} \quad n \times n \text{ matrix} \\ \Theta(\lambda + 1) \Theta(1 - \lambda) \Theta(y - x) \left(J_1(\lambda) \Theta(x - \lambda) + J_2(\lambda) \Theta(\lambda - y) \right), \\ \text{where } J_1(\lambda), J_2(\lambda) \text{ are arbitrary } C^{\infty} \text{ matrices, such that } J_1(-1) = J_2(1) = 0 \end{cases}$$

I is an identity matrix,

g_0 is an arbitrary constant matrix,

is a solution of the Ernst equation (13).

The proof is based on the following lemmas 1 and 2.

Lemma. 1 Let

$$\nu(x, y, \lambda_0) = 2 \left(I + \sum_{n=1}^{\infty} (0, \quad I) \int Z^{-}_{n,n-1} Z^{-}_{n-1,n-2} \dots Z^{-}_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} d\lambda_n d\lambda_{n-1} \dots d\lambda_1 \right), \tag{15}$$

then

$$\begin{aligned} & \frac{1}{2}g^{-1}(x, y)\nu(x, y, \lambda_0) = \\ & = g_0^{-1} \left(I + \sum_{n=1}^{\infty} \int (0, \quad I) Z^+_{n,n-1} Z^+_{n-1,n-2} \dots Z^+_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} d\lambda_n d\lambda_{n-1} \dots d\lambda_1 \right), \end{aligned} \quad (16)$$

where $g(x, y)$ is the series (14).

The proof is by induction.

Lemma. 2 Let $g(x, y)$ and $\nu(x, y, \lambda_0)$ be given by the series (14) and (15), respectively. Then

$$\begin{aligned} \frac{\partial}{\partial x} \nu(x, y, \lambda_0) &= \frac{1}{2} \left(1 + \left(\frac{\lambda_0 - y}{\lambda_0 - x} \right)^{\frac{1}{2}} \right) g_x g^{-1} \nu(x, y, \lambda_0), \\ \frac{\partial}{\partial y} \nu(x, y, \lambda_0) &= \frac{1}{2} \left(1 + \left(\frac{\lambda_0 - x}{\lambda_0 - y} \right)^{\frac{1}{2}} \right) g_y g^{-1} \nu(x, y, \lambda_0) \end{aligned} \quad (17)$$

as formal series.

Proof.

We define the generalized function x^λ , and x_+^λ , $\lambda \neq -1, -2, \dots$ as in [10]. The integrals like

$$\int_0^\infty f(x) x^{-\frac{3}{2}} dx$$

should be understood as the analytic continuation of the integral

$$\int_0^\infty f(x) x^\lambda dx.$$

The generalized functions x^λ have the properties

$$\begin{aligned} \frac{d}{dx}(x^\lambda) &= \lambda x^{\lambda-1}, \quad \lambda \neq 0 \\ x x^\lambda &= x^{\lambda+1}. \end{aligned} \quad (18)$$

All the kernels in (14) and (15) and their derivatives are well-defined as generalized functions. Using the properties (18) of these generalized functions, we will work with the kernels as with usual functions.

Since the formulas (17) are symmetric in x, y , it is enough to prove the first equation in (17) only.

Using Lemma 1 to compute the product $\frac{1}{2}g^{-1}(x, y)\nu(x, y, \lambda_0)$, the identity we need to prove in degree n in Z reads:

$$\begin{aligned} & \frac{\partial}{\partial x} (0, \quad I) Z_{n,n-1}^- Z_{n-1,n-2}^- \dots Z_{1,0}^- \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \\ & - \left(1 + \left(\frac{\lambda_0 - y}{\lambda_0 - x} \right)^{\frac{1}{2}} \right) \sum_m \frac{\partial}{\partial x} \left((0, \quad I) Z_{n,n-1}^- Z_{n-1,n-2}^- \dots Z_{m+1,m}^- \begin{pmatrix} -\frac{F_m}{p_m} \\ \frac{p_m}{J_m} \\ -\frac{J_m}{p_m} \end{pmatrix} \right) \cdot \\ & (0, \quad I) Z_{m-1,m-2}^+ Z_{m-2,m-3}^+ \dots Z_{1,0}^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0 \end{aligned} \tag{19}$$

where

$$p_k \equiv p(\lambda_k) \equiv (\lambda - x)^{\frac{1}{2}}(\lambda - y)^{\frac{1}{2}}$$

, and the block-diagonal matrices Z are defined in (14).

Applying the Leibnits rule and using the identities

$$\begin{aligned} & \frac{\partial}{\partial x} Z_{k,k-1}^\pm = -\frac{1}{2} \begin{pmatrix} \frac{F_k}{p_k(\lambda_k - x)} \\ \frac{J_k}{p_k(\lambda_k - x)} \end{pmatrix} \left(\left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}}, \quad \pm 1 \right) \\ & \frac{\partial}{\partial x} \begin{pmatrix} -\frac{F_k}{p_k} \\ \frac{J_k}{p_k} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \frac{F_k}{p_k(\lambda_k - x)} \\ \frac{J_k}{p_k(\lambda_k - x)} \end{pmatrix} \\ & \begin{pmatrix} \frac{F_m}{p_m} \\ \frac{p_m}{J_m} \\ \frac{J_m}{p_m} \end{pmatrix} (0, \quad I) = \frac{1}{2} (Z_{m,m-1}^+ - Z_{m,m-1}^-), \end{aligned}$$

the left-hand side of (19) can be rewritten as

$$\sum_k (0, I) Z^-_{n,n-1} Z^-_{n-1,n-2} \dots Z^-_{k+1,k} \cdot W_k$$

where

$$\begin{aligned} W_k &= \left(\frac{\partial}{\partial x} Z^-_{k,k-1} \right) Z^-_{k-1,k-2} \dots Z^-_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \\ &\quad - (1 + (\frac{\lambda_0 - y}{\lambda_0 - x})^{\frac{1}{2}}) \sum_m \left(\frac{\partial}{\partial x} Z^-_{k,k-1} \right) \cdot \\ &\quad \cdot Z^-_{k-1,k-2} \dots Z^-_{m+1,m} \begin{pmatrix} -\frac{F_m}{J_m} \\ -\frac{p_m}{p_m} \end{pmatrix} (0, I) Z^+_{m-1,m-2} Z^+_{m-2,m-3} \dots Z^+_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \\ &\quad + \frac{1}{2} (1 + (\frac{\lambda_0 - y}{\lambda_0 - x})^{\frac{1}{2}}) \begin{pmatrix} \frac{F_k}{J_k} \\ \frac{p_k(\lambda_k - x)}{p_k(\lambda_k - x)} \end{pmatrix} Z^+_{k-1,k-2} Z^+_{k-2,k-3} \dots Z^+_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \\ &= \frac{1}{2} \begin{pmatrix} \frac{F_k}{p_k(\lambda_k - x)} \\ \frac{J_k}{p_k(\lambda_k - x)} \end{pmatrix} (W^-_{k-1} - W^+_{k-1}), \end{aligned}$$

where

$$W^\pm_k = (1 \pm (\frac{\lambda_0 - y}{\lambda_0 - x})^{\frac{1}{2}}) \left(\mp \left(\frac{\lambda_k - y}{\lambda_k - x} \right)^{\frac{1}{2}}, 1 \right) Z^\pm_{k,k-1} Z^\pm_{k-1,k-2} Z^\pm_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Define also

$$V^\pm_k = (1 \pm (\frac{\lambda_0 - y}{\lambda_0 - x})^{\frac{1}{2}}) \left(1, \mp \left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}} \right) Z^\pm_{k,k-1} Z^\pm_{k-1,k-2} Z^\pm_{1,0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We will prove by induction that

$$\begin{aligned} W^+_k &= W^-_k \\ V^+_k &= V^-_k. \end{aligned} \tag{20}$$

For $k = 1$, the identities (20) can be checked easily. Using the identities,

$$\begin{aligned}
& \left(\mp \left(\frac{\lambda_k - y}{\lambda_k - x} \right)^{\frac{1}{2}}, \quad 1 \right) Z^{\pm}_{k,k-1} = \\
& = \frac{F_k}{(\lambda_k - \lambda_{k-1})} \frac{p_{k-1}}{p_k} \left(1, \quad \mp \left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}} \right) + \frac{J_k}{(\lambda_k - \lambda_{k-1})} \frac{(\lambda_{k-1} - x)}{(\lambda_k - x)} \left(\mp \left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}}, \quad 1 \right); \\
& \left(1, \quad \mp \left(\frac{\lambda_k - y}{\lambda_k - x} \right)^{\frac{1}{2}} \right) Z^{\pm}_{k,k-1} = \\
& \frac{J_k}{(\lambda_k - \lambda_{k-1})} \frac{p_{k-1}}{p_k} \left(1, \quad \mp \left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}} \right) + \frac{F_k}{(\lambda_k - \lambda_{k-1})} \frac{(\lambda_{k-1} - x)}{(\lambda_k - x)} \left(\mp \left(\frac{\lambda_{k-1} - y}{\lambda_{k-1} - x} \right)^{\frac{1}{2}}, \quad 1 \right) \\
& \tag{21}
\end{aligned}$$

we can reduce an expression with product of k matrices Z in (20) to a similar expression with product of only $k - 1$ matrices Z , which proves the lemma by induction assumption.

The proof of the Theorem follows from Lemmas 1,2 and from the fact that the compatibility condition for equations (17) gives the Ernst equation (13). Since we have shown that with $g(x, y)$ given by (14) both equations of (17) are satisfied, the compatibility conditions is fulfilled automatically, and thus $g(x, y)$ solves the equation (13).

Lemma. 3 *Let*

$$\begin{aligned}
& F(\lambda) = 0 \\
& J(\lambda) := \Theta(\lambda + 1)\Theta(1 - \lambda)\Theta(y - x) \left(J_1(\lambda)\Theta(x - \lambda) + J_2(\lambda)\Theta(\lambda - y) \right), \\
& g_0 = I
\end{aligned} \tag{22}$$

where Θ is the Heaviside function. Then $g(x, y)$ given by (14) is real. If J_1 and J_2 are symmetric matrices, then $g(x, y)$ is a symmetric matrix.

Proof With our choice of the branch of the square root, the reality condition is obvious. The fact that we get a symmetric matrix can be proved by induction.

We can ensure that the determinat of the matrix $g(x, y)$ is $(x - y)^2$ by specifying $g(x, 1)$ and $g(-1, y)$ and finding recursively $J_1(\lambda)$ and $J_2(\lambda)$, using

the defining relation (14) and the inversion formula for the Abel transform, discussed in the linear case.

3 Quantum version.

The equations (4), (7) are the Euler-Lagrange equations for the action

$$S = \frac{1}{\alpha} \int (det g)^{\frac{1}{2}} \left(\varphi_{uv} + Tr \left(g_u g^{-1} g_v g^{-1} \right) \right) du \wedge dv, \quad (23)$$

This action, up to a total derivative, is just the Einstein action, specialized for metrics which are constant along the integral lines of the Killing vector fields and written for metrics in the gauge

$$ds^2 = g_{ab}(u, v) dx^a dx^b - 2(det(g))^{-\frac{1}{2}} \exp\left(\frac{1}{4}\varphi(u, v)\right) du dv.$$

This Lagrangian may actually describe some specially arranged scattering experiment, probably the fluctuations in the background of two colliding gravitational plane waves, where the measurements are averaged in the normal to the collision plane directions by the detector, so only the constant in those directions mode survive.

In any event, this Lagrangian gives an interesting two dimensional model, worth studying in the quantum case, especially since firstly it has something to do with gravity, and secondly it is not that infinitely far from being treatable by conventional methods, as the 4 dimensional gravity is. It looks essentially as the chiral field Lagrangian, with the field g taking values not in the unitary group, but in symmetric positive-definite matrices, namely, in 2×2 matrices for reduction from 4 dimensions. (If we would take g with values in the $SU(n)$ group instead, the determinant of g is automatically 1, which trivially solves the wave equation, and our Lagrangian becomes just the principal chiral field Lagrangian of $SU(n)$; such model was solved exactly in the quantum case, [2].) Our Lagrangian has a global $SL(2, \mathbb{R})$ symmetry,

$$g \rightarrow h^T g h, \quad h \in SL(2, \mathbb{R}), \quad (24)$$

corresponding to basis changes in the Killing directions. The power counting for this Lagrangian works in the same way as for the chiral field, or a σ model, so the issue of renormalizability is quite clear. However, the renormalization

flow is different, compared to the unitary chiral field, and, essentially, is more like a renormalization flow for the $SL(2, \mathbb{R})$ chiral field (such model, with the WZW term added, was solved recently in [4]; it is curious however that for symmetric 2×2 matrices the WZW term $Tr(dgg^{-1} \wedge^* 3)$ is identically zero). The rough argument here is to use the σ -model language and to say that the β -function is proportional to the Ricci tensor, which is no longer positive-definite, and therefore it's not an asymptotically free case, as it was for a unitary chiral field.

When reducing from 4 dimensions, the Killing directions should be thought of as, say, compactified on a torus, then the integral over those non-interesting coordinates gives simply the area of the torus, and we get a Planck constant in front which is of order $(l_{planck}/l_\perp)^2$, where l_\perp is a typical length in directions normal to the collision plane; this is really a pretty small number, thus we expect that the loop expansion should be reliable, and we will consider the quantum corrections to the Lagrangian in one loop only. Our treatment is similar to that for a σ model, [6], [3], [5]. It could be very helpful to use the technique developed for the chiral field model, like in [1], for the case of non-unitary or non-compact group.

Unlike the case of the σ -model or principal chiral field, there is a natural additive structure for symmetric matrices, so we just expand around, say, one of the classical solutions g, φ found in the beginning of the paper,

$$\begin{aligned}\tilde{g}(u, v) &= g(u, v) + h(u, v) \\ \varphi(u, v) &= \varphi(u, v) + \phi(u, v),\end{aligned}$$

and keep only the quadratic terms in fluctuations, to get

$$S = \frac{1}{\alpha} S_0 + \langle \nabla_u \xi, \nabla_v \xi \rangle + \xi R \xi, \quad (25)$$

where S_0 is, accidentally, zero on equations of motion, and $\xi = (h_{11}, h_{22}, h_{12}, \phi)^T$, and $\langle \rangle$ and R can be expressed in terms of the components of the background fields g, φ , as follows

$$\begin{aligned}\langle \xi \eta \rangle &\equiv \xi^T m \eta, \\ m &= (det(g))^{-\frac{3}{2}} \begin{pmatrix} (g_{22})^2 & (g_{12})^2 & -2g_{22}g_{12} & -\frac{1}{4}g_{22}det(g) \\ (g_{12})^2 & (g_{11})^2 & -2g_{11}g_{12} & -\frac{1}{4}g_{11}det(g) \\ -2g_{22}g_{12} & -2g_{11}g_{12} & 2(g_{11}g_{22} + (g_{12})^2) & \frac{1}{2}g_{12}detg \\ -\frac{1}{4}g_{22}det(g) & -\frac{1}{4}g_{11}det(g) & \frac{1}{2}g_{12}detg & 0 \end{pmatrix} \quad (26)\end{aligned}$$

Although m does not transform as a metric under linear transformations of ξ , we use it to define an $SL(2, \mathbb{R})$ -invariant scalar product, (Minkowski signature), with $SL(2, \mathbb{R})$ acting as (24). We can find an orthogonal reper for this scalar product $e_{(\alpha)}$, $\alpha = 1, 2, 3, 4$, of the form

$$(e_{(1)}, e_{(2)}, e_{(3)}, e_{(4)}) = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

, such that, for g positive-definite, and

$$\langle e_{(\alpha)} e_{(\beta)} \rangle = \eta \equiv \text{diag}(1, 1, 1, -1); \quad (27)$$

$\nabla = \partial + a$ is defined as follows:

$$\begin{aligned} \nabla_u h &= \partial_u h - h g^{-1} g_u - g_u g^{-1} h + \frac{1}{2} \text{Tr}(g^{-1} h) g_u - \frac{1}{4 \langle e_{(4)} e_{(4)} \rangle} (\det(g))^{\frac{1}{2}} \text{Tr}(g^{-1} h g^{-1} g_u) e_{(4)} \\ \nabla_u \chi &= \partial_u \chi \end{aligned} \quad (28)$$

$$\begin{aligned} \xi \ R \ \xi &= \det(g)^{\frac{1}{2}} \left(\right. \\ &\left(-\frac{1}{8} (\text{Tr}(g^{-1} h))^2 - \frac{1}{4} \text{Tr}(g^{-1} h g^{-1} h) \right) \text{Tr}(g_u g^{-1} g_v g^{-1}) + \\ &+ \left(\frac{1}{8} (\text{Tr}(g^{-1} h))^2 - \frac{1}{4} \text{Tr}(g^{-1} h g^{-1} h) \right) \varphi_{uv} + \\ &+ \text{Tr}(g^{-1} h) \text{Tr}(g_\mu g^{-1} g^\mu g^{-1} h g^{-1}) - \text{Tr}(g_u g^{-1} h g^{-1} g_v g^{-1} h g^{-1}) - \\ &\left. - \frac{1}{16} (\det(g))^{\frac{1}{2}} \text{Tr}(g_u g^{-1} h g^{-1}) \text{Tr}(g_v g^{-1} h g^{-1}) \right) \end{aligned} \quad (29)$$

Substituting $\xi = \chi^\alpha e_{(\alpha)}$ in the Lagrangian we get

$$S = \partial \chi \partial \chi + A \chi A \chi + 2 \partial \chi A \chi + \chi (e \ R \ e) \chi, \quad (30)$$

where

$$A = \tilde{e}(\partial + a)e, \quad (31)$$

with a defined in (28), and \tilde{e} is dual to the reper e , (27),

$$\tilde{e}_i^{(\alpha)} e_{(\alpha)}^j = \delta_i^j.$$

We now switch to the Euclidean version, so u, v should be now z, \bar{z} , and the metric η is Euclidean, and use the dimensional regularization to compute Feynman diagrams.

There are two logarithmically divergent Feynmann diagrams with the background A fields, contributing to the effective action, the bubble and the loop with 2 χ propagators and 2 $\partial\chi A\chi$ vertices. The divergent contribution of the antisymmetric in internal indices part of A coming from these two diagrams cancels one another, due to the fact that

$$\begin{aligned} \int \frac{d^d k}{k^2 + \varepsilon^2} &\sim \Gamma(1 - \frac{d}{2}) \varepsilon^{d-2} = \frac{1}{2-d} + \text{regular} \\ \int \frac{k_\mu k_\nu}{(k^2 + \varepsilon^2)((k-q)^2 + \varepsilon^2)} d^d k &\sim \frac{1}{2} \delta_{\mu\nu} \Gamma(1 - \frac{d}{2}) \int_0^1 dt (\varepsilon^2 + t(1-t)q^2)^{\frac{d-2}{2}} + \text{regular} = \\ &= \frac{1}{2} \delta_{\mu\nu} \frac{1}{2-d} + \text{regular}, \end{aligned} \tag{32}$$

and the factors of 2 and minus signs comes just right for those diagrams to cancel one another; we have put in a mass to cut off in the infrared, but nothing depend from it, and there many ways to fix the infrared problems, so we are not concerned with that. For the symmetric in internal indices part of A , the loop diagram is not divergent, and the bubble diagram is, so there is a total contribution of the A terms to the effective action. Adding it to the contribution of the bubble diagram with $(e R e)$, we get the following correction to the effective action in one loop:

$$\frac{\mu^{d-2}}{2-d} \left(\frac{1}{2} (A_{\alpha_1 \alpha_2} A_{\alpha_1 \alpha_2} - A_{\alpha_1 \alpha_2} A_{\alpha_2 \alpha_1}) + e^i_{(\alpha)} R_{ij} e^j_{(\alpha)} \right) \tag{33}$$

Those correction terms have two 'world-sheet' derivatives in them, so they can be thought of as a correction to what is the metric in the σ -model language, and therefore, the meaning of the above computation is in that it shows what is the replacement in our case for the well-known fact about σ models, that the β function is proportional to the Ricci tensor. In the σ model, the renormalizations can be organized to correct the metric and to produce the dilaton and tachion potentials; however, I have not found a convenient way to keep track of the corrections for this model.

4 Discussion

We have constructed explicitly local solutions of free Einstein equations, in the case when there are $d-2$ commuting Killing vector fields. We have, in

fact, all local solutions, since we can satisfy the appropriate initial data. It will be very interesting to understand the global properties of these solutions, and what kind of singularities they develop. To achieve that, we firstly need a better control over the convergence of the series. For integrable equations like the nonlinear Schrodinger, we can prove that the perturbation series is convergent, and we are working on the convergence for the Ernst. Secondly, our solutions are written in particular coordinates, which make the computation easy. However, to understand the singularities, we need to formulate everything invariantly.

We proposed a two dimensional model, describing quantum fluctuations of the metric, constant along the Killing directions. This model is interesting just by itself, since it is close to the chiral field model, and thus apparently treatable, but has a different renormalization flow in the ultraviolet, and, most probably, is not asymptotically free. We have studied the renormalization in one loop. This model is definitely not an easy one, and more have to be done to understand where it actually flows.

There is another reason of why the model is interesting, since it might describe some specially arranged scattering experiment, when only those two dimension we keep matters. If so, this will give an insight for gravity in 4 dimension, which is not treatable by standart field theory methods.

Our 2 dimensional model can be also looked at as some kind of string theory, but with the world-sheet having quite concrete interpretation in terms of the target space geometry.

It will be interesting to have the supersymmetric version of this model.

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